Algebra Qualifying Exam II (August 2023)

You have 120 minutes to complete this exam.

1. (10 points) Let R be an integral domain containing a field K as a subring. Suppose that R is a finite dimensional vector space over K under the ring multiplication. Prove that R is a field.

2. (10 points) Let A be a ring with identity. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence of left A-modules. Prove that if L and N are both finitely generated, then M is also finitely generated.

3. (10 points) Let $A = \mathbb{Q}[x, x^{-1}]$ be the ring of Laurent polynomials. Find an A-module which is torsion free but not free.

4. (10 points) Let A be a nontrivial commutative ring with identity. Suppose that every ideal in A is free as an A-module. Prove that A is a principal ideal domain.

5. (10 points) Let A be a ring and let $\alpha \in A$ be an element such that $\alpha^2 = \alpha$. Prove that the left ideal

$$(\alpha) := \{r\alpha \mid r \in A\}$$

is a projective A-module.

6. (10 points) Compute the Tor group

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/15, \mathbb{Z}/6).$$